Quadratic Equations

When we **solve** an **equation** like $x + 7 = 12$, we remember that we must subtract 7 from both sides of the equal sign.

$$x + 7 = 12$$
$$x + 7 - 7 = 12 - 7 \quad \text{subtract 7 from both sides}$$
$$x = 5$$

That leaves us with $x = 5$. We know that 5 is the only **solution** or **value** that can replace x and make the $x + 7 = 12$ true.

If $x + 7 = 12$, and
$x = 5$ is true, then
$5 + 7 = 12$.

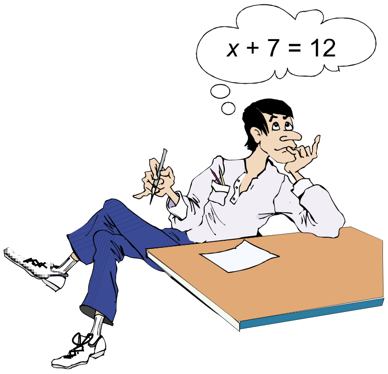
Suppose you have an *equation* that looks like $(x + 7)(x - 3) = 0$. This means there are two numbers, one in each set of parentheses, that when multiplied together, have a **product** of 0. What kinds of numbers can be multiplied and equal 0?

Look at the following options.

$2 \times -2 = -4$ $\qquad$ $\frac{1}{5} \times 5 = 1$ $\qquad$ $-\frac{4}{7} \times \frac{7}{4} = -1$

The only way for numbers to be multiplied together with a result of zero is if one of the numbers is a 0.

$a \times 0 = 0$



Looking back at $(x + 7)(x - 3) = 0$, we understand that there are two **factors**, $(x + 7)$ and $(x - 3)$. The only way to multiply them and get a *product* of 0 is if one of them is equal to zero.

This leads us to a way to *solve* the equation. Since we don't know which of the terms equals 0, we cover all the options and assume either could be equal to zero.

If $x + 7 = 0$,
then $x = -7$.

If $x - 3 = 0$,
then $x = 3$.

We now have two options which could replace x in the original equation and make it true. Let's replace x with -7 and 3, one at a time.

$$\begin{aligned} (x + 7)(x - 3) &= 0 \\ (-7 + 7)(-7 - 3) &= 0 \\ (0)(-10) &= 0 \\ 0 &= 0 \end{aligned} \qquad \begin{aligned} (x + 7)(x - 3) &= 0 \\ (3 + 7)(3 - 3) &= 0 \\ (10)(0) &= 0 \\ 0 &= 0 \end{aligned}$$

Therefore, because either *value* of x gives us a true statement, we see that the **solution set** for $(x + 7)(x - 3) = 0$ is {-7, 3}.

Now you try the items in the following practice.

Factoring to Solve Equations

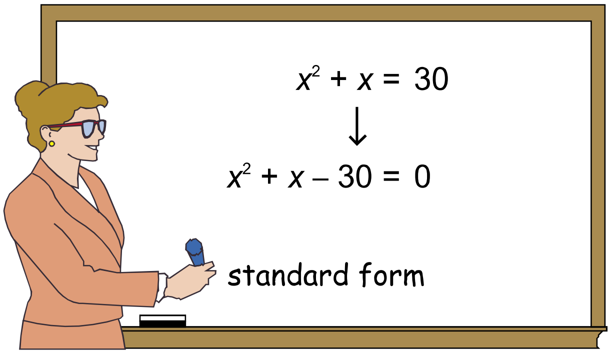
Often, equations are not given to us in **factored form** like those on the previous pages. Looking at $x^2 + x = 30$, we notice the x^2 **term** which tells us this is a **quadratic equation** (an equation in the form $ax^2 + bx + c = 0$). This term also tells us to be on the lookout for two answers in our *solution set*.

You may solve this problem by trial and error. However, we can also solve $x^2 + x = 30$ using a format called **standard form (of a quadratic equation).** This format is written with the *terms* in a special order:

- the x^2 term first
- then the x-term
- then the numerical term followed by = 0.

For our original equation,

$$x^2 + x = 30 \quad \leftarrow \text{put in } \textit{standard form}$$
$$x^2 + x - 30 = 30 - 30 \quad \leftarrow \text{subtract 30 from both sides}$$
$$x^2 + x - 30 = 0$$



Now that we have the proper format, we can factor the quadratic **polynomial**.

Remember: Factoring expresses a *polynomial* as the product of **monomials** and polynomials.

Example 1

Solve by factoring

$x^2 + x - 30 = 0$	← factor
$(x + 6)(x - 5) = 0$	

Set each factor equal to 0

If $x + 6 = 0$, then	← **zero product property**
$x = -6$.	← add -6 to each side
If $x - 5 = 0$, then	← zero product property
$x = 5$.	← add 5 to each side

Therefore, the solution set is {-6, 5}.

Example 2

Solve by factoring

$x^2 = 5x - 4$	← put in standard form
$x^2 - 5x + 4 = 0$	← factor
$(x - 4)(x - 1) = 0$	

Set each factor equal to 0

If $x - 4 = 0$, then	← zero product property
$x = 4$.	← add -4 to each side
If $x - 1 = 0$, then	← zero product property
$x = 1$.	← add -1 to each side
{1, 4}	← write the solution set

Now it's your turn to practice on the following page.

Solving Word Problems

We can also use the processes on pages 715-716 and 719-720 to solve word problems. Let's see how.

Example 1

Two **consecutive** (in order) **positive integers** (integers greater than zero) have a product of 110. Find the integers.

$$\begin{aligned} \text{let the 1}^{\text{st}}\text{ integer} &= x \\ \text{and the 2}^{\text{nd}}\text{ integer} &= x + 1 \end{aligned}$$

$$\begin{aligned} x(x + 1) &= 110 \\ x^2 + x &= 110 \\ x^2 + x - 110 &= 0 \\ (x - 10)(x + 11) &= 0 \\ x - 10 &= 0 \quad \text{or} \quad x + 11 = 0 \\ x &= 10 \quad \text{or} \quad x = -11 \end{aligned}$$

Since the problem asked for *positive integers*, we must eliminate -11 as an answer. Therefore, the two integers are $x = 10$ and $x + 1 = 11$.

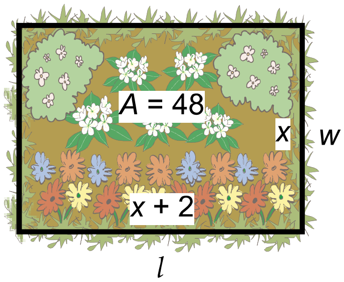
Remember: Integers are the numbers in the set {… , -4, -3, -2, -1, 0, 1, 2, 3, 4, …}.

Example 2

Billy has a garden that is 2 feet longer than it is wide. If the **area (*A*)** of his garden is 48 square feet, what are the dimensions of his garden?

If we knew the **width (*w*)**, we could find the **length (*l*)**, which is 2 feet longer. Since we don't know the *width*, let's represent it with x. The *length* will then be $x + 2$.

$$\begin{aligned} \text{width} &= x \\ \text{length} &= x + 2 \end{aligned}$$



The *area* (*A*) of a **rectangle** can be found using the **formula** length (l) times width (w).

$$\begin{aligned} A &= lw \\ A &= 48 \end{aligned}$$

So,
$$\begin{aligned} x(x + 2) &= 48 \\ x^2 + 2x &= 48 \\ x^2 + 2x - 48 &= 0 \\ (x + 8)(x - 6) &= 0 \\ x + 8 = 0 \quad &\text{or} \quad x - 6 = 0 \\ x = -8 \quad &\text{or} \quad x = 6 \end{aligned}$$

A garden cannot be -8 feet long, so we must use only the 6 as a value for x.

So, the width of the garden is 6 feet and the length is 8 feet.

Using the Quadratic Formula

Sometimes an equation seems difficult to factor. When this happens, you may need to use the **quadratic formula**. Remember that quadratic equations use the format below.

$$ax^2 + bx + c = 0$$

The *quadratic formula* uses the information from the equation and looks like the following.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Compare the equation with the formula and notice how all the same letters are just in different places.

Let's see how the quadratic formula is used to solve the following equation.

In the equation, $a = 4$, $b = 1$, and $c = -5$. These values are **substituted** into the quadratic formula below.

$$4x^2 + x - 5 = 0$$ ← original equation

$$\boxed{a = 4 \quad b = 1 \quad c = -5}$$ ← values

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ ← quadratic formula

$$x = \frac{-1 \pm \sqrt{(1)^2 - 4(4)(-5)}}{2(4)}$$ ← values a = 4, b = 1, and c = -5 *substituted*

$$x = \frac{-1 \pm \sqrt{1 - (-80)}}{8}$$ ← simplify

$$x = \frac{-1 \pm \sqrt{81}}{8}$$

$$x = \frac{-1 \pm 9}{8}$$

$$x = \frac{-1 + 9}{8} \quad \text{or} \quad x = \frac{-1 - 9}{8}$$

$$x = \frac{8}{8} \quad \text{or} \quad x = \frac{-10}{8}$$

$$x = 1 \quad \text{or} \quad x = \frac{-5}{4}$$ ← the solution set is $\{1, \frac{-5}{4}\}$

Remember: The symbol ± means plus or minus. Therefore, ± means we have two factors. One is found by adding and the other by subtracting.

Let's look at another example.

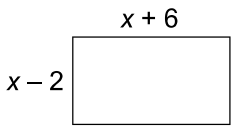
$2x^2 + 5x + 3 = 0$	← original equation
$a = 2 \quad b = 5 \quad c = 3$	← values
$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	← quadratic formula
$x = \frac{-5 \pm \sqrt{(5)^2 - 4(2)(3)}}{2(2)}$	← values *a* = 2, *b* = 5, and *c* = 3 *substituted*
$x = \frac{-5 \pm \sqrt{25 - 24}}{4}$	← simplify
$x = \frac{-5 \pm \sqrt{1}}{4}$	
$x = \frac{-5 \pm 1}{4}$	
$x = \frac{-5 + 1}{4}$ or $x = \frac{-5 - 1}{4}$	
$x = \frac{-4}{4}$ or $x = \frac{-6}{4}$	
$x = -1$ or $x = \frac{-3}{2}$	← the solution set is $\{-1, \frac{-3}{2}\}$

Often your answer will not **simplify** all the way to a **fraction** or *integer*. Some answers will look like $\frac{-3 \pm \sqrt{13}}{6}$. You can check your work by using a graphing calculator or another advanced-level calculator.

We can even use the quadratic formula when solving word problems. However, before you start using the quadratic formula, it is important to remember to put the equation you are working with in the correct format. The equation must look like the following.

$$ax^2 + bx + c = 0$$

Look at this example.



If a rectangle has an area of 20 and its dimensions are as shown, find the actual length and width of the rectangle.

$(x + 6)(x - 2) = 20$	⟵ set up the equation
$x^2 + 4x - 12 = 20$	⟵ **FOIL**—First, Outside, Inside, Last
$x^2 + 4x - 32 = 0$	⟵ format ($ax^2 + bx + c = 0$)
$(x + 8)(x - 4) = 0$	⟵ factor
$x + 8 = 0$ ***or*** $x - 4 = 0$	⟵ solve
$x = -8$ ***or*** $x = 4$	
$x \neq -8$ because that would result in negative lengths.	⟵ first check to see *if* solutions are reasonable

Remember: The symbol ≠ means is ***not*** equal to.

$x + 6$ ⟶ $4 + 6 = 10$	⟵ then check answer by replacing x with 4
$x - 2$ ⟶ $4 - 2 = 2$	

The length and width of the rectangle are 10 and 2.